An extended relativistic quantum oscillator for $S = 1$ particles

1998 J. Phys. A: Math. Gen. 31 3867

(http://iopscience.iop.org/0305-4470/31/16/014)

J. Phys. A: Math. Gen. **31** (1998) 3867–3874. Printed in the UK PII: S0305-4470(98)85635-7

An extended relativistic quantum oscillator for $S = 1$ particles

Y Nedjadi†, S Ait-Tahar† and R C Barrett‡

† Nuclear and Astrophysics Laboratory, University of Oxford, Keble Rd, Oxford OX1 3RH, UK
‡ Department of Physics, University of Surrey, Guildford GU2 5XH, UK

Received 3 July 1997, in final form 25 November 1997

Abstract. We introduce the *extended* Duffin–Kemmer–Petiau (DKP) oscillator obtained by combining two relativistic quantum oscillator models. In a study analogous to Kukulin, Loyola and Moshinsky's work on extended Dirac oscillators, we investigate whether this extended version has oscillator shells controllably independent from the spin–orbit coupling. This extended DKP oscillator is found to be *exactly* solvable for natural parity states. We calculate and discuss both the natural- and unnatural-parity eigenspectra of its spin-1 representation.

1. Introduction

The study of relativistic quantum harmonic oscillators has been of increasing interest in recent years in investigations concerned, for instance, with their covariance and CPT properties [1], group-theoretical properties [2], hidden supersymmetric nature [3], geometric quantization and coherent state formulation [4], generalization to particles with arbitrary spin [5], properties at the first-quantized level and various other issues [6].

Here we shall consider an extension of the recently proposed relativistic Duffin–Kemmer–Petiau (DKP) oscillator [7]. This extension is constructed by a procedure analogous to that developed by Kukulin, Loyola and Moshinsky for the extended Dirac oscillator [8]. This procedure consists of combining a Lorentz tensor external field linear in $\boldsymbol{r}$ together with a timelike Lorentz vector one quadratic in $\boldsymbol{r}$.

In the non-relativistic limit, the $S = 1$ DKP oscillator presents, in addition to the usual three-dimensional harmonic oscillations, a spin–orbit coupling whose strength is one half that obtained from a Dirac oscillator with the same frequency [7]. Yet the energy level splittings this spin–orbit component produces are rather large, of the order of magnitude of the oscillator energy levels themselves.

In [8], Kukulin *et al* showed that, for the Dirac equation, their particular combination of two oscillator models made it possible to generate an extended version of the Dirac oscillator whose spin–orbit coupling is controllably independent from oscillator shells. The interest in such an extension lies in the fact that the structure of hadron spectra is sensitive to the strength of the spin–orbit coupling.

The focus of this paper is to investigate whether an analogous extension can be realized, and exactly solved, in the case of a DKP relativistic quantum oscillator for $S = 1$ particles.

2. The DKP oscillator

Before introducing its *extended* version, consider the basic features of the DKP oscillator [7]. For a free vector boson of mass m, the relativistic DKP equation [9] is

$$(c\boldsymbol{\beta}\cdot\boldsymbol{p}+mc^2)\psi=\mathrm{i}\hbar\beta^0\frac{\partial\psi}{\partial t} \tag{2.1}$$

where the internal variables β^μ ($\mu=0,1,2,3$) satisfy the commutation relation

$$\beta^\mu\beta^\nu\beta^\lambda+\beta^\lambda\beta^\nu\beta^\mu=g^{\mu\nu}\beta^\lambda+g^{\nu\lambda}\beta^\mu. \tag{2.2}$$

In the spin-1 representation, β^μ are 10×10 matrices while the dynamical state ψ is a ten-component spinor.

For the Lorentz tensor external potential which we introduce with the non-minimal substitution

$$\boldsymbol{p}\to\boldsymbol{p}-\mathrm{i}m\omega\eta^0\boldsymbol{r} \tag{2.3}$$

where ω is the oscillator frequency and $\eta^0=2(\beta^0)^2-1$, the DKP equation for the system is

$$[c\boldsymbol{\beta}\cdot(\boldsymbol{p}-\mathrm{i}m\omega\eta^0\boldsymbol{r})+mc^2]\psi=\mathrm{i}\hbar\beta^0\frac{\partial\psi}{\partial t}. \tag{2.4}$$

In the spin-1 representation of equation (2.4), the dynamical state ψ is chosen as the ten-component spinor

$$\psi(\boldsymbol{r})=\begin{pmatrix}\mathrm{i}\varphi(\boldsymbol{r})\\ \boldsymbol{A}(\boldsymbol{r})\\ \boldsymbol{B}(\boldsymbol{r})\\ \boldsymbol{C}(\boldsymbol{r})\end{pmatrix}\qquad\text{with } \boldsymbol{A}\equiv\begin{pmatrix}A_1\\A_2\\A_3\end{pmatrix}\qquad\boldsymbol{B}\equiv\begin{pmatrix}B_1\\B_2\\B_3\end{pmatrix}\qquad\boldsymbol{C}\equiv\begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix} \tag{2.5}$$

so that, for stationary states, the equation of motion equation (2.4) decomposes into

$$\begin{aligned}mc^2\varphi&=\mathrm{i}c\boldsymbol{p}^-\cdot\boldsymbol{B}\\ mc^2\boldsymbol{A}&=E\boldsymbol{B}-c\boldsymbol{p}^+\wedge\boldsymbol{C}\\ mc^2\boldsymbol{B}&=E\boldsymbol{A}+\mathrm{i}c\boldsymbol{p}^+\varphi\\ mc^2\boldsymbol{C}&=-c\boldsymbol{p}^-\wedge\boldsymbol{A}\end{aligned} \tag{2.6}$$

where $\boldsymbol{p}^\pm=\boldsymbol{p}\pm\mathrm{i}m\omega\boldsymbol{r}$. Since $\boldsymbol{A}$ is the three-component spinor analogous to the Dirac upper component, we seek the wave equation for $\boldsymbol{A}$. It is straightforward to eliminate φ, $\boldsymbol{B}$ and $\boldsymbol{C}$ in favour of $\boldsymbol{A}$ so that one gets

$$\begin{aligned}(E^2-m^2c^4)\boldsymbol{A}&=[c^2(\boldsymbol{p}^2+m^2\omega^2\boldsymbol{r}^2)-3\hbar\omega mc^2-2\hbar\omega mc^2\boldsymbol{L}\cdot\boldsymbol{s}]\boldsymbol{A}\\ &\quad-\frac{1}{m^2}\boldsymbol{p}^+\{\boldsymbol{p}^-\cdot[\boldsymbol{p}^+\wedge(\boldsymbol{p}^-\wedge\boldsymbol{A})]\}\end{aligned} \tag{2.7}$$

where $\boldsymbol{L}$ is the orbital angular momentum and $\boldsymbol{s}$ the 3×3 spin-1 operator. Using $E=\varepsilon+mc^2$ and the non-relativistic limit $\varepsilon\ll mc^2$, the fourth term in equation (2.7) becomes negligible, since it is of the order of $1/m^3$, so that the wave equation for $\boldsymbol{A}$ can be written

$$\varepsilon\boldsymbol{A}\simeq\left[\frac{\boldsymbol{p}^2}{2m}+\frac{1}{2}m\omega^2\boldsymbol{r}^2-\frac{3}{2}\hbar\omega-\hbar\omega\boldsymbol{L}\cdot\boldsymbol{s}\right]\boldsymbol{A} \tag{2.8}$$

which characterizes the usual harmonic oscillator in addition to a spin–orbit coupling, absent for scalar DKP bosons, of strength $-\hbar\omega$. Note that the strength of this coupling is half the one obtained from the Dirac oscillator [10].

Since the spin-1 representation of equation (2.4) leads to the usual three-dimensional (3D) oscillator, in the non-relativistic limit, we refer to the system it describes as the DKP oscillator.

3. The extended DKP oscillator

We now turn to the extended model in which the quadratic Lorentz vector piece $m\Omega^2 r^2 \beta^0 \mathcal{P}$ is included as an additional external field in equation (2.4), i.e.

$$[c\boldsymbol{\beta}\cdot(\boldsymbol{p} - \mathrm{i}m\omega\eta^0\boldsymbol{r}) + m\Omega^2\boldsymbol{r}^2\beta^0\mathcal{P} + mc^2]\psi = \mathrm{i}\hbar\beta^0\frac{\partial\psi}{\partial t}. \tag{3.1}$$

$\mathcal{P}$ is a projection operator which picks out the four upper components of the DKP spinor. As will be shown below, the logic underlying this prescription lies in the fact that for $\omega = 0$ the non-relativistic limit of (3.1) yields a harmonic oscillator without spin–orbit coupling (only a quadratic interaction term remains). Alternative prescriptions, such as including $\Omega^2\boldsymbol{r}^2$ as a Lorentz scalar term for instance, would be inadequate. It can be simply shown that in the non-relativistic limit they would yield the usual 3D oscillator potential in addition to three types of tensor terms built up from $\boldsymbol{s}$, $\boldsymbol{r}$ and $\boldsymbol{p}$. This would violate the necessary requirement that relativistic generalizations of quantum oscillators should recover the usual 3D oscillator in the non-relativistic limit.

For arbitrary ω and Ω frequencies, using the same analytical procedure as above decomposes the equation of motion (3.1) into

$$\begin{aligned}
mc^2\varphi &= \mathrm{i}c\boldsymbol{p}^-\cdot\boldsymbol{B}\\
mc^2\boldsymbol{A} &= E\boldsymbol{B} - c\boldsymbol{p}^+\wedge\boldsymbol{C}\\
mc^2\boldsymbol{B} &= (E - m\Omega^2\boldsymbol{r}^2)\boldsymbol{A} + \mathrm{i}c\boldsymbol{p}^+\varphi\\
mc^2\boldsymbol{C} &= -c\boldsymbol{p}^-\wedge\boldsymbol{A}.
\end{aligned} \tag{3.2}$$

A lengthy but otherwise straightforward calculation of the relevant wave equation for $\boldsymbol{A}$ leads to

$$\begin{aligned}
(E^2 - m^2c^4)\boldsymbol{A} = &\left[c^2\left(\boldsymbol{p}^2 + m^2\left(\omega^2 + \frac{E}{mc^2}\Omega^2\right)\boldsymbol{r}^2\right) - 3\hbar\omega mc^2 - 2\hbar\omega mc^2\boldsymbol{L}\cdot\boldsymbol{s}\right]\boldsymbol{A}\\
&-\frac{1}{m^2}\boldsymbol{p}^+\{\boldsymbol{p}^-\cdot[\boldsymbol{p}^+\wedge(\boldsymbol{p}^-\wedge\boldsymbol{A})]\}
\end{aligned} \tag{3.3}$$

which, in the non-relativistic limit, reduces to

$$\varepsilon\boldsymbol{A} \simeq \left[\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m(\omega^2 + \Omega^2)\boldsymbol{r}^2 - \frac{3}{2}\hbar\omega - \hbar\omega\boldsymbol{L}\cdot\boldsymbol{s}\right]\boldsymbol{A}. \tag{3.4}$$

This designates the usual harmonic oscillator (with frequency $(\omega^2 + (E/mc^2)\Omega^2)^{1/2}$) combined with a $\hbar\omega$ strong spin–orbit coupling. In this extended oscillator model, unlike the simple DKP oscillator in section 2, the spin–orbit splitting can controllably be decoupled from the oscillator shells. In the $\omega = 0$ limit, the harmonic oscillator has frequency Ω and there is no spin–orbit coupling.

4. Solution of the extended DKP oscillator problem

We now seek to calculate the complete solution to the extended DKP oscillator. The total angular momentum $\boldsymbol{J}$ can be shown to be conserved in which case the general eigenfunction one can use takes the form [11]

$$\psi_{JM}(\boldsymbol{r}) = \frac{1}{r}\begin{pmatrix} \mathrm{i}\phi_{nJ}(r)Y_{JM}(\Omega) \\ \sum_L F_{nJL}(r)\mathbf{Y}^M_{JL1}(\Omega) \\ \sum_L G_{nJL}(r)\mathbf{Y}^M_{JL1}(\Omega) \\ \sum_L H_{nJL}(r)\mathbf{Y}^M_{JL1}(\Omega) \end{pmatrix}. \tag{4.1}$$

Putting ψ_{JM} into equation (3.1) results in ten coupled radial differential equations which can be decoupled into two sets associated with $(-1)^J$ and $(-1)^{J+1}$ parities. We call the $(-1)^J$ solutions natural-parity (or magnetic-like) states while we refer to the $(-1)^{J+1}$ solutions as unnatural-parity (or electric-like) states. With the notation

$$R_{nJJ}(r) = R_0 \qquad R_{nJJ\pm1}(r) = R_{\pm1} \qquad R \equiv F, G, H \tag{4.2}$$

and the definition $\alpha_J = ((J+1)/(2J+1))^{1/2}$ and $\zeta_J = (J/(2J+1))^{1/2}$, the set associated with $(-1)^J$ parity is

$$(E - m\Omega^2 r^2)F_0 = mc^2 G_0 \tag{4.3a}$$

$$\hbar c\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J+1}{r} + \frac{m\omega r}{\hbar}\right)F_0 = -\frac{1}{\zeta_J}mc^2 H_1 \tag{4.3b}$$

$$\hbar c\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J}{r} + \frac{m\omega r}{\hbar}\right)F_0 = -\frac{1}{\alpha_J}mc^2 H_{-1} \tag{4.3c}$$

$$-\zeta_J\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J+1}{r} - \frac{m\omega r}{\hbar}\right)H_1 - \alpha_J\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J}{r} - \frac{m\omega r}{\hbar}\right)H_{-1} = \frac{1}{\hbar c}\left(mc^2 F_0 - EG_0\right). \tag{4.3d}$$

For unnatural-parity states, the radial differential equations are coupled in the following way:

$$\hbar c\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J+1}{r} - \frac{m\omega r}{\hbar}\right)H_0 = -\frac{1}{\zeta_J}(mc^2 F_1 - EG_1) \tag{4.4a}$$

$$\hbar c\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J}{r} - \frac{m\omega r}{\hbar}\right)H_0 = -\frac{1}{\alpha_J}(mc^2 F_{-1} - EG_{-1}) \tag{4.4b}$$

$$-\zeta_J\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J+1}{r} + \frac{m\omega r}{\hbar}\right)F_1 - \alpha_J\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J}{r} + \frac{m\omega r}{\hbar}\right)F_{-1} = \frac{1}{\hbar c}mc^2 H_0 \tag{4.4c}$$

$$\hbar c\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J+1}{r} - \frac{m\omega r}{\hbar}\right)\phi = \frac{1}{\alpha_J}((E - m\Omega^2 r^2)F_1 - mc^2 G_1) \tag{4.4d}$$

$$\hbar c\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J}{r} - \frac{m\omega r}{\hbar}\right)\phi = -\frac{1}{\zeta_J}((E - m\Omega^2 r^2)F_{-1} - mc^2 G_{-1}) \tag{4.4e}$$

$$-\alpha_J\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J+1}{r} + \frac{m\omega r}{\hbar}\right)G_1 + \zeta_J\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J}{r} + \frac{m\omega r}{\hbar}\right)G_{-1} = \frac{1}{\hbar c}mc^2\phi. \tag{4.4f}$$

The nature of the coupling between the different radial equations reflects the way in which the tensor and the vector external fields mix the components of the DKP spinor between themselves.

4.1. Natural-parity states

The exact solution for the magnetic-like states is obtained by eliminating G_0 and $H_{\pm1}$ in equations (4.3), i.e.

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(E^2 - m^2c^4)}{(\hbar c)^2} + \frac{m\omega}{\hbar} - \frac{m^2\omega'^2 r^2}{\hbar^2} - \frac{J(J+1)}{r^2}\right)F_0(r) = 0 \tag{4.5}$$

where $\omega'^2 = \omega^2 + (E/mc^2)\Omega^2$, so that the associated eigenvalues can simply be shown to obey the eigenvalue equation

$$\frac{1}{2mc^2}(E_{N,J}^2 - m^2c^4) = \left(N + \frac{3}{2}\right)\hbar\left(\omega^2 + \frac{E_{N,J}}{mc^2}\Omega^2\right)^{1/2} - \frac{1}{2}\hbar\omega. \tag{4.6}$$

This is a fourth-order equation in $E_{N,J}$; N is the principal quantum number $N = 2n + J$ (n is the radial quantum number).

In the limit where $\Omega \simeq 0$, as would be expected, the eigenenergy reduces to that of the basic DKP oscillator [7], that is

$$\frac{1}{2mc^2}(E_{N,J}^2 - m^2c^4) = (N+1)\hbar\omega. \tag{4.7}$$

For very low ω and for $\omega \ll \Omega$ frequencies, the eigenenergy equation yields a fourth-degree equation for E:

$$\frac{1}{2mc^2}(E_{N,J}^2 - m^2c^4) = \left(N + \frac{3}{2}\right)\hbar\Omega\left(\frac{E_{N,J}}{mc^2}\right)^{1/2} - \frac{1}{2}\hbar\omega. \tag{4.8}$$

If we now consider the non-relativistic limit of equation (4.6), the oscillator energy levels turn out to be

$$\begin{aligned}\varepsilon_{N,J} \simeq {} & \left(\left(N + \frac{3}{2}\right)\hbar(\omega^2 + \Omega^2)^{1/2} - \frac{1}{2}\hbar\omega\right) \\ & \times\left(1 + \frac{1}{2mc^2}\left(N + \frac{3}{2}\right)\hbar(\omega^2 + \Omega^2)^{1/2}\frac{\Omega^2}{\Omega^2 + \omega^2}\right).\end{aligned} \tag{4.9}$$

The exact F_0 eigenfunctions satisfying equation (4.5) take a form similar to that given in equation (32) of [7] and the radial components G_0 and $H_{\pm 1}$ can be simply deduced from equations (4.3*a*)–(4.3*c*).

4.2. *Unnatural-parity states*

Now for the eigenspectrum of electric-like states, the relevant coupled radial differential equations one needs to solve are those in equations (4.4). An analytic solution does exist for $J = 0$ states. In this case it is possible to transform equations (4.4) into

$$\begin{aligned}&\frac{d^2}{dr^2}F_{-1}(r) + \left(\frac{(E^2 - m^2c^4)}{(\hbar c)^2} + \frac{m\omega}{\hbar} - \frac{m^2\omega'^2 r^2}{\hbar^2}\right)F_{-1}(r) = 0 \\ &H_0(r) = -\frac{1}{mc^2}\left(\frac{d}{dr} + m\omega r\right)F_{-1}(r) \\ &G_{-1}(r) = \frac{1}{mc^2}(E - m\Omega^2 r^2)F_{-1}(r)\end{aligned} \tag{4.10a}$$

and

$$\begin{aligned}&\frac{d^2}{dr^2}G_1(r) + \left(\frac{(E^2 - m^2c^4)}{(\hbar c)^2} - \frac{m\omega}{\hbar} - \frac{m^2\omega'^2 r^2}{\hbar^2} - \frac{2}{r^2}\right)G_1(r) = 0 \\ &\phi(r) = -\frac{1}{mc^2}\left(\frac{d}{dr} + \frac{1}{r} + m\omega r\right)G_1(r) \\ &F_1(r) = \frac{E}{mc^2}G_1(r).\end{aligned} \tag{4.10b}$$

The $\{H_0, F_{-1}, G_{-1}\}$ set of radial wavefunctions comes out as decoupled from the set $\{\phi, F_1, G_1\}$. Clearly F_{-1} and G_1 obey harmonic-oscillator-type radial equations for which there are two sets of 0^- oscillator shells.

The first set of 0^- states, those for which $\{\phi, F_1, G_1\}$ are identically zero, comprises oscillator shells which have the same energies as the natural-parity 0^+ levels since F_{-1} obeys the same eigenequation as F_0 with $J = 0$ (see equation (4.5)). In the non-relativistic limit, the binding energies are described by

$$\varepsilon_{N,0^-} \simeq \left(\left(N + \frac{3}{2}\right)\hbar(\omega^2 + \Omega^2)^{1/2} - \frac{1}{2}\hbar\omega\right) \times \left(1 + \frac{1}{2mc^2}\left(N + \frac{3}{2}\right)\hbar(\omega^2 + \Omega^2)^{1/2}\frac{\Omega^2}{\Omega^2 + \omega^2}\right). \tag{4.11a}$$

The corresponding radial eigenfunction F_{-1} can be simply expressed in terms of associated Laguerre polynomials (see equation (32) of [7]) while G_{-1} and H_0 can be trivially specified using equation (4.10*a*).

The other class of 0^- oscillator levels—those whose radial eigencomponents $\{H_0, F_{-1}, G_{-1}\}$ vanish—coincide with the magnetic-like 1^+ oscillator shells because G_1 satisfies the same oscillator eigenequation as F_0 with $J = 1$ and $\hbar\omega$ lower zero-point energy. In the non-relativistic limit, the 0^- eigenenergies are

$$\varepsilon_{N,0^-} \simeq \left(\left(N + \frac{3}{2}\right)\hbar(\omega^2 + \Omega^2)^{1/2} + \frac{1}{2}\hbar\omega\right) \times \left(1 + \frac{1}{2mc^2}\left(N + \frac{3}{2}\right)\hbar(\omega^2 + \Omega^2)^{1/2}\frac{\Omega^2}{\Omega^2 + \omega^2}\right). \tag{4.11b}$$

The radial eigenfunctions $\{G_1, F_1, \phi\}$ can be simply obtained as above.

In the case of unnatural-parity states with $J > 0$, there is no obvious non-trivial particular solution to equations (4.4). These equations can be transformed into ($\hbar = c = 1$)

$$\frac{d^2\phi}{dr^2} + P_\phi(r)\frac{d\phi}{dr} + \left(Q_\phi(r) - \frac{J(J+1)}{r^2}\right)\phi = \Xi_{H_0}(r)H_0 \tag{4.12a}$$

$$\frac{d^2H_0}{dr^2} + P_{H_0}(r)\frac{dH_0}{dr} + \left(Q_{H_0}(r) - \frac{J(J+1)}{r^2}\right)H_0 = \Xi_\phi\phi \tag{4.12b}$$

with

$$P_\phi = -\frac{\Gamma'}{\Gamma} \qquad Q_\phi = E^2 - m^2 - m^2\omega'^2r^2 - 3m\omega + \left(\frac{1}{r} + m\omega r\right)\frac{\Gamma'}{\Gamma} \tag{4.12c}$$

$$P_{H_0}(r) = -\frac{\Gamma'}{\Gamma} \qquad Q_{H_0}(r) = E^2 - m^2 - m^2\omega'^2r^2 - m\omega + m\omega r\frac{\Gamma'}{\Gamma} \tag{4.12d}$$

and

$$\Xi_{H_0} = \sqrt{J(J+1)}\left(\frac{\Lambda'\Gamma}{mr} + 2(E - V)\omega\right) \qquad \Xi_\phi = \sqrt{J(J+1)}\left(-\frac{E\Gamma'}{mr\Gamma} + 2E\omega\right) \tag{4.12e}$$

while

$$\Gamma(r) = (E^2 - m^2 - EV) \qquad \Lambda(r) = \frac{1}{\Gamma}(E - V) \qquad V = m\Omega^2r^2. \tag{4.12f}$$

The prime on Γ or Λ designates a derivative with respect to r. Once equations (4.11a) and (4.11b) are solved, the remaining unnatural-parity radial wavefunctions are simply given by

$$\begin{pmatrix} F_1 \\ G_1 \end{pmatrix} = \frac{1}{\Gamma}\begin{pmatrix} \alpha_J E & \zeta_J m \\ \alpha_J m & \zeta_J(E-V) \end{pmatrix}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{J+1}{r} - m\omega r\right)\begin{pmatrix} \phi \\ H_0 \end{pmatrix} \quad (4.13a)$$

$$\begin{pmatrix} F_{-1} \\ G_{-1} \end{pmatrix} = \frac{1}{\Gamma}\begin{pmatrix} -\zeta_J E & \alpha_J m \\ -\zeta_J m & \alpha_J(E-V) \end{pmatrix}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{J}{r} - m\omega r\right)\begin{pmatrix} \phi \\ H_0 \end{pmatrix}. \quad (4.13b)$$

In this new form, we have reduced the problem of solving equations (4.4a)–(4.4f) to that of solving a pair of coupled second-order differential equations which seem to represent two *coupled* 3D harmonic oscillators, albeit with a rather complicated coupling.

For the frequency $\Omega = 0$, as required, these equations reduce to the unnatural-parity state radial equations (33a)–(33d) of the basic DKP oscillator [7]. On the other hand, for $\omega = 0$, the radial equations associated only with the Lorentz timelike vector oscillator interaction have structural similarities to those found for electric states of a DKP boson in a Coulomb field [11], or of the Breit wave equation for two equal-mass fermions interacting via a Coulomb potential [12]. These equations are known not to have analytical solutions.

Disregarding the coupling terms on the right-hand sides of equations (4.12a) and (4.12b), both the homogeneous equations in ϕ and H_0 have three regular singularities (at $r = 0, -r_0, r_0$ with $r_0 = ((E^2 - m^2)/Em\Omega^2)^{1/2}$) and an irregular singularity at infinity (a fourth-order pole).

These equations cannot be transformed into confluent hypergeometric forms, which involve only one regular and one irregular singularity, while non-closed-form solutions in terms of power series expansions valid in the ranges between the different singularities would be very difficult to obtain (fourth-order coupled recurrence relations). Approximate and asymptotic solutions shall not be considered in this paper.

5. Conclusion

We have constructed an extended DKP oscillator by combining a Lorentz tensor external field linear in $\boldsymbol{r}$ with a timelike Lorentz vector potential quadratic in the separation.

In the non-relativistic limit, the DKP equation of motion leads to the usual harmonic oscillator with a spin–orbit coupling of the Thomas form. However, unlike the basic DKP oscillator case, here the oscillator shells are controllably independent from the spin–orbit splittings since they are specified by different frequency parameters.

We have shown that this relativistic oscillator is exactly solvable for magnetic-like states. In the case of unnatural-parity states, only 0^- states have analytic eigensolutions. For higher angular momentum states, although the eigenproblem can be reduced to that of two non-trivially coupled 3D harmonic oscillators, it does not seem to admit to known analytic solutions.

Acknowledgment

The financial support of EPSRC through GR/H53648 is gratefully acknowledged.

References

[1] Moreno M and Zentella A 1989 *J. Phys. A: Math. Gen.* **22** L821
[2] Moshinsky M, Loyala G and Villegas C 1991 *J. Math. Phys.* **32** 373
[3] Benitez J, Martinez y Romero R P, Nunez-Yepez H N and Salas-Brito A L 1990 *Phys. Rev. Lett.* **64** 1643

Moreno M, Martinez R and Zentella A 1990 *Mod. Phys. Lett.* A **5** 949
Rubakov V A and Spiridinov V P 1988 *Mod. Phys. Lett.* A **3** 1337
[4] Aldaya V and Guerrero J 1995 *J. Math. Phys.* **36** 3191
[5] Moshinsky M and del Sol Mesa A 1996 *J. Phys. A: Math. Gen.* **29** 4217
[6] Han D and Wolf K B (eds) 1995 *2nd Int. Workshop on Harmonic Oscillators* (Maryland: NASA)
[7] Nedjadi Y and Barrett R C 1994 *J. Phys. A: Math. Gen.* **27** 4301
[8] Kukulin V I, Loyola G and Moshinsky M 1991 *Phys. Lett.* **158A** 19
[9] Petiau G 1936 *Acad. R. Belg. Cl. Sci. Mem. Collect.* **16** No 2
Kemmer N 1938 *Proc. R. Soc.* **166** 127
Kemmer N 1939 *Proc. R. Soc.* A **173** 91
Duffin R J 1939 *Phys. Rev.* **54** 1114
[10] Moshinsky M and Szczepaniak A 1989 *J. Phys. A: Math. Gen.* **22** L817
[11] Nedjadi Y and Barrett R C 1994 *J. Math. Phys.* **35** 4517
[12] Malenfant J 1988 *Phys. Rev.* D **38** 3295